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ON THE VARIATIONAL METHOD IN THE THEORY OF CONTACT PROBLEMS FOR NONLINEARLY ELASTIC LAMINATED BODIES

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Variational formulation of contact problems for systems of deformable bodies with nonlinear stress-strain relationship is proposed and substantiated. It is assumed that at small deformations the body may separate in a part of the initial contact surface, but increase of the contact areas is excluded. Layered foundations and laminated cylinders have these properties, hence the considered system of bodies may be conveniently called laminated body. The problem is formulated as one of minimization of the total functional on a convex closed set of Sobolev's functional space. Conditions of existence and uniqueness of solution of the variational problem are determined. An equivalent formulation in the form of a variational inequality is also proposed.

The problem of separation of an elastic body from a rigid half-space (the problem of Signorini) was investigated in [1] using the variational method. The variational theory of contact between a rigid stamp and a nonlinearly elastic body was presented in [2]. Variational formulation of the problem of contact between several deformable bodies with allowance for an initial gap between these was investigated in [3]. Known methods of investigation of laminated bodies [4—7] assumed a linear relation between stress and strain and depend to a considerable extent on the shape of layers.

The variational method of investigation of laminated bodies is virtually independent of the assumption of linearity, homogeneity, and isotropy, and unrelated to the shape of layers. Moreover no prior information on the actual areas of contact is required. Numerical methods, such as the method of finite elements [8] can be effectively used and substantiated.

1. Statement of the problem. Let us consider a system of N bodies occupying regions $\Omega_1, \Omega_2, \ldots, \Omega_N$ of the three-dimensional space E, bounded by smooth surfaces $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$. We denote by Γ_{mn} the common boundary of bodies Ω_m and Ω_n in the initial undeformed state, and assume that for each m the set Γ_{mn} is nonempty for at least one n. We also assume that for small deformations the surfaces of actual contact between bodies (layers) does not increase over the initial contact areas.

Let $x = (x_1, x_2, x_3) \in E$, $u(x) = (u_1(x), u_2(x), u_3(x))$ be the vector of small displacements, and $\varepsilon_{ij}(x)$, $\sigma_{ij}(x)$ the tensors of small deformations and of stresses, respectively.

Let us formulate the assumed conditions of contact of layers Ω_m and Ω_n at the boundary Γ_{mn}. We denote by $\nu(x)$ the unit vector external relative to Ω_n and normal to Γ_{mn}, and introduce the normal and tangential components of displacement vector and of the stress vector on surface Γ_{mn}

$$u_\nu = u_i \nu_i, \quad (u_\tau)_i = u_i - u_\nu \nu_i, \quad \sigma_\nu = \sigma_{ij}\nu_i\nu_j, \quad (\sigma_\tau)_i = \sigma_{ij}\nu_j - \sigma_\nu \nu_i$$

In what follows the superscripts denote the oridinal number of the layer to which a particular quantity relates.

The following conditions must be satisfied at points of surface Γ_{mn}:

$$\begin{aligned} \sigma_\nu^{(n)}(x) &= -\sigma_\nu^{(m)}(x) = \sigma_\nu(x) \\ \sigma_\tau^{(n)}(x) &= -\sigma_\tau^{(m)}(x) = \sigma_\tau(x), \quad \forall x \in \Gamma_{mn} \end{aligned} \tag{1.1}$$

Three types of interaction between layers Ω_m and Ω_n are considered on surface Γ_{mn}:

bonded layers

$$u^{(m)}(x) = u^{(n)}(x), \quad \forall x \in \Gamma_{mn} \tag{1.2}$$

separation can take place but relative slippage of layers is excluded

$$\begin{aligned} &u_\nu^{(m)}(x) \geqslant u_\nu^{(n)}(x), \quad u_\tau^{(m)}(x) = u_\tau^{(n)}(x) \\ &\sigma_\nu(x) \leqslant 0, \quad \sigma_\nu(x)\,[u_\nu^{(m)}(x) - u_\nu^{(n)}(x)] = 0, \quad \forall x \in \Gamma_{mn} \end{aligned} \tag{1.3}$$

separation and relative slippage of layers may occur

$$\begin{aligned} &u_\nu^{(m)}(x) \geqslant u_\nu^{(n)}(x), \quad \sigma_\tau(x) = 0 \\ &\sigma_\nu(x) \leqslant 0, \quad \sigma_\nu(x)\,[u_\nu^{(m)}(x) - u_\nu^{(n)}(x)] = 0, \quad \forall x \in \Gamma_{mn} \end{aligned} \tag{1.4}$$

The conditions of form $u_\nu^{(m)}(x) \geqslant u_\nu^{(n)}(x)$ postulate mutual impenetrability of layers, and conditions (1.3) and (1.4) take into account that $\sigma_\nu(x) = 0$ when separation takes place at point $x \in \Gamma_{mn}$, and $\sigma_\nu(x) \leqslant 0$ applies in the opposite case.

It is assumed that surface

$$\Gamma_k - \bigcup_m \Gamma_{km} \quad (k = 1, 2, \ldots, N)$$

may consist of three parts: $\Gamma_k^u, \Gamma_k^X, \Gamma_k^c$. On part Γ_k^u we specify displacements and on Γ_k^X stresses

$$u(x) = U^{(k)}(x), \quad \forall x \in \Gamma_k^u \tag{1.5}$$

$$\sigma_{ij}(x)\,\nu_j(x) = X_i^{(k)}(x), \quad \forall x \in \Gamma_k^X \tag{1.6}$$

On part Γ_k^c layer Ω_k is subjected to the action of the stamp whose boundary is defined by the equation $\Psi_k(x) = 0$, and outside the stamp $\Psi_k(x) > 0$.

On surface Γ_k^c we assume the following conditions [2]:

$$\begin{aligned} &\Psi_k(x) + u(x)\,\mathrm{grad}\,\Psi_k(x) \geqslant 0, \quad \sigma_\tau(x) = 0 \\ &\sigma_\nu(x) \leqslant 0, \quad \sigma_\nu(x)\,[\Psi_k(x) + u(x)\,\mathrm{grad}\,\Psi_k(x)] = 0, \ \forall x \in \Gamma_k^c \end{aligned} \tag{1.7}$$

which determine the normal (frictionless) contact between the rigid and the deformed body.

Each layer is, furthermore, subjected to the action of mass forces $\rho_i^{(k)}(x)$, $x \in \Omega_k$, $k = 1, 2, \ldots, N$.

The relation between stresses and strains is assumed nonlinear, and defined by [9]

$$\begin{gathered}\sigma_{ij} = \lambda_k(x)\,\vartheta\delta_{ij} + 2\mu_k(x)\,\varepsilon_{ij} - 2\mu_k(x)\,\omega_k(x, e_u)\,e_{ij} \\ \vartheta = \varepsilon_{ii}, \quad e_{ij} = \varepsilon_{ij} - {}^1/_3\,\vartheta\delta_{ij}, \quad e_u = ({}^2/_3\,e_{ij}e_{ij})^{1/2} \\ \Lambda \geqslant \lambda_k(x) \geqslant \lambda > 0, \quad M \geqslant \mu_k(x) \geqslant \mu > 0\end{gathered} \tag{1.8}$$

We introduce the function of strain energy density

$$W_k(x, \varepsilon_{ij}) = \int \sigma_{ij}(x, \varepsilon_{ij})\,d\varepsilon_{ij} = \frac{1}{2}\lambda_k(x)\,\vartheta^2 + \mu_k(x)\,\varepsilon_{ij}\varepsilon_{ij} - 3\mu_k(x)\int_0^{e_u} s\omega_k(x, s)\,ds \tag{1.9}$$

Function $\omega_k(x, s)$ is assumed to be such that the following conditions are satisfied:

1) $W_k(x, \varepsilon_{ij})$ is a convex function continuously differentiable with respect to ε_{ij} for any $x \in \Omega_k$, consequently, the inequality

$$W_k(x, \varepsilon_{ij}') - W_k(x, \varepsilon_{ij}) \geqslant \frac{\partial W_k(x, \varepsilon_{ij})}{\partial \varepsilon_{ij}}(\varepsilon_{ij}' - \varepsilon_{ij}) = \sigma_{ij}(\varepsilon_{ij}' - \varepsilon_{ij}) \tag{1.10}$$

is satisfied [10];

2) there exists such $\alpha_k > 0$ that

$$(1 - \alpha_k)\left[\frac{1}{2}\lambda_k(x)\,\vartheta^2 + \mu_k(x)\,\varepsilon_{ij}\varepsilon_{ij}\right] \geqslant 3\mu_k(x)\int_0^{e_u} s\omega_k(x, s)\,ds \tag{1.11}$$

The indicated requirements are satisfied when function $\omega_k(x, s)$ satisfies conditions

$$\begin{gathered}0 \leqslant \omega_k(x, s) \leqslant \partial(s\omega_k(x, s)) / \partial s \leqslant \alpha_k < 1 \\ \partial\omega_k(x, s) / \partial s \geqslant 0, \quad \forall x \in \Omega_k\end{gathered}$$

The problem consists of the determination of the displacement vector u_i, and of tensors of strain ε_{ij} and stress σ_{ij} that satisfy the equations of equilibrium, Cauchy's relations, nonlinear formulas (1.8), and the conditions (1.1) — (1.7) imposed on stresses and displacements.

2. Variational formulation of the problem. We introduce for each layer Ω_k the Sobolev's space $H^1(\Omega_k)$ of vector functions $u^{(k)}(x) = (u_1^{(k)}(x), u_2^{(k)}(x), u_3^{(k)}(x))$ which has generalized first derivatives and is square summable. We determine in $H^1(\Omega_k)$ the scalar product

$$(u^{(k)}, v^{(k)})_{H^1(\Omega_k)} = \int_{\Omega_k} (u_i^{(k)}v_i^{(k)} + u_{i,j}^{(k)}v_{i,j}^{(k)})\,d\Omega$$

Let us consider the basic space $H^1(\Omega)$ consisting of vector functions $u(x)$

determined on $\Omega = \bigcup_k \Omega_k$ as follows: if $x \in \Omega_k$, then $u(x) = u^{(k)}(x)$. As the scalar product in $H^1(\Omega)$ we take

$$(u, v)_{H^1(\Omega)} = \sum_{k=1}^{N} (u^{(k)}, v^{(k)})_{H^1(\Omega_k)} \tag{2.1}$$

For brevity of notation we shall use besides $H^1(\Omega)$ also the symbol H.

We impose on discontinuities along Γ_{mn} constraints which correspond to conditions defined in (1.2) – (1.4) and separate the following subsets of functions:

$$V_1^{(mn)} = \{v \mid v \in H,\ v^{(m)}(x) = v^{(n)}(x),\ \forall x \in \Gamma_{mn}\}$$
$$V_2^{(mn)} = \{v \mid v \in H,\ v_\nu^{(m)}(x) \geqslant v_\nu^{(n)}(x),\ v_\tau^{(m)}(x) = v_\tau^{(n)}(x),\ \forall x \in \Gamma_{mn}\}$$
$$V_3^{(mn)} = \{v \mid v \in H,\ v_\nu^{(m)}(x) \geqslant v_\nu^{(n)}(x),\ \forall x \in \Gamma_{mn}\}$$

We also separate the subsets of functions $v \in H$ that satisfy the kinematic conditions on Γ_k^c and specified on Γ_k^u by conditions

$$V_c^{(k)} = \{v \mid v \in H,\ \Psi_k(x) + v(x) \operatorname{grad} \Psi_k(x) \geqslant 0,\ \forall x \in \Gamma_k^c\}$$
$$V_u^{(k)} = \{v \mid v \in H,\ v(x) = U^{(k)}(x),\ \forall x \in \Gamma_k^u\}$$

Each of the introduced sets is convex and closed. Finally, we construct the set V of vector functions that satisfy all conditions imposed on displacements

$$V = \bigcap_{m,n} V^{(mn)} \bigcap_k V_c^{(k)} \bigcap_k V_u^{(k)}$$

where $V^{(mn)}$ is one of sets $V_1^{(mn)}, V_2^{(mn)}, V_3^{(mn)}$. Set V is convex and closed, being the intersection of a finite number of convex and closed sets.

We present the formal derivation of the variational principle for displacements in the theory of laminated bodies, assuming that the requirements for regularity of functions are satisfied. Let $u(x) \in V$ be the actual displacements of the body Ω and $\varepsilon_{ij}(x)$, $\sigma_{ij}(x)$ the real values of the strain and stress tensors. We introduce the admissible displacements $v(x) \in V$ and related strains $\varepsilon_{ij}'(x) = {}^1/_2\ (v_{i,j}(x) + v_{j,i}(x))$, and consider the equilibrium of layer Ω_k, replacing the action of remaining layers by corresponding forces on surface Γ_{kn} $(n = 1, 2, \ldots, N)$. Applying to the integral

$$\int_{\Omega_k} \frac{\partial}{\partial x_j} [\sigma_{ij}(v_i - u_i)]\, d\Omega$$

the Ostrogradskii – Gauss formula and taking into account that for actual stresses equilibrium conditions are satisfied, we obtain

$$\int_{\Omega_k} \sigma_{ij}(\varepsilon_{ij}' - \varepsilon_{ij})\, d\Omega - \int_{\Gamma_k^X} X_i^{(k)}(v_i - u_i)\, d\Gamma - \int_{\Omega_k} \rho_i^{(k)}(v_i - u_i)\, d\Omega - \tag{2.2}$$
$$\sum_{n=1}^{N} \int_{\Gamma_{kn}} [\sigma_\nu(v_\nu - u_\nu) + \sigma_\tau(v_\tau - u_\tau)]\, d\Gamma - \int_{\Gamma_k^c} \sigma_\nu(v_\nu - u_\nu)\, d\Gamma = 0$$
$$(k = 1, 2, \ldots, N)$$

For each of the three types of interaction of surface Γ_{mn} $(m, \ n = 1, 2, \ldots, N)$ the following conditions are satisfied:

$$\sigma_\nu(x)\,[v_\nu^{(n)}(x) - u_\nu^{(n)}(x) - v_\nu^{(m)}(x) + u_\nu^{(m)}(x)] \geqslant 0 \tag{2.3}$$
$$\sigma_\tau(x)\,[v_\tau^{(n)}(x) - u_\tau^{(n)}(x) - v_\tau^{(m)}(x) + u_\tau^{(m)}(x)] = 0, \quad \forall x \in \Gamma_{mn}$$

At points of surfaces $\Gamma_k{}^c$ we have the relationships

$$\sigma_\nu(x)\,[v_\nu(x) - u_\nu(x)] \geqslant 0, \qquad \forall x \in \Gamma_k{}^c \ (k = 1, 2, \ldots, N) \tag{2.4}$$

Adding (2.2) for $k = 1, 2, \ldots, N$ and taking into account (1.10), (2.3), and (2.4) we obtain

$$\sum_{k=1}^{N}\Big[\int_{\Omega_k} W_k(x, \varepsilon_{ij})\,d\Omega - \int_{\Gamma_k{}^X} X_i^{(k)} u_i\,d\Gamma - \int_{\Omega_k} \rho_i^{(k)} u_i\,d\Omega\Big] \leqslant$$
$$\sum_{k=1}^{N}\Big[\int_{\Omega_k} W_k(x, \varepsilon_{ij}')\,d\Omega - \int_{\Gamma_k{}^X} X_i^{(k)} v_i\,d\Gamma - \int_{\Omega_k} \rho_i^{(k)} v_i\,d\Omega\Big], \quad \forall v \in V$$

Thus the following variational principle has been proved: among all displacements $v \in V$ the minimum of functional

$$J(v) = {}^1/_2\, B(v, v) - \Pi(v) - F(v) \tag{2.5}$$
$$B(u, v) = \sum_{k=1}^{N}\int_{\Omega_k} [\lambda_k(x)\,\vartheta\vartheta' + 2\mu_k(x)\,\varepsilon_{ij}\varepsilon_{ij}']\,d\Omega$$
$$\Pi(v) = \sum_{k=1}^{N}\int_{\Omega_k} 3\mu_k(x) \int_0^{e_u(v)} s\,\omega_k(x, s)\,ds\,d\Omega$$
$$F(v) = \sum_{k=1}^{N}\Big[\int_{\Omega_k} \rho_i^{(k)} v_i\,d\Omega + \int_{\Gamma_k{}^X} X_i^{(k)} v_i\,d\Gamma\Big]$$

corresponds to actual displacements.

On the basis of the proved above principle we formulate the variational problem

$$\inf_{v \in V} J(v) \tag{2.6}$$

which may also be formulated as the equivalent problem of solving the variational inequality [11]

$$X(W(u), v - u) \geqslant F(v - u), \quad \forall v \in V \tag{2.7}$$
$$X(W(u), v - u) = \sum_{k=1}^{N}\int_{\Omega_k} \frac{\partial W_k(x, \varepsilon_{ij})}{\partial \varepsilon_{ij}}(\varepsilon_{ij}' - \varepsilon_{ij})\,d\Omega$$

Let us clarify the conditions under which the variational problem (2.6) has a meaning. We assume that $\Psi_k \in C^1$, $U_i^{(k)} \in H^{1/2}(\Gamma_k{}^u)$, $X_i^{(k)} \in H^{-1/2}(\Gamma_k{}^X)$, $\rho_i^{(k)} \in L^2(\Omega_k)$. Functions $X_i^{(k)}$ and $\rho_i^{(k)}$ may be piecewise continuous, and functions $U_i^{(k)}$ piecewise differentiable in respective regions. Then, using the trace theorem [12] and taking into account previous assumptions about functions

$W_k(x, \varepsilon_{ij})$, it is possible to show that the integrals in (2.5) exist for all $v \in H$ and that set V is nonempty.

The above reasoning is valid if the surface of each layer is smooth. However the variational problem can be formulated in all cases, if the functional $J(v)$ is determinate on H and set V nonempty. Problems of existence of solutions are investigated in Sect. 3 with fairly general assumptions about surface regularity, introduced in [13].

3. The existence and uniqueness of solutions of the variational problem. Let us consider the subset R of rigid displacements whose elements are such displacements $v \in H$ with which each layer moves as a rigid body

$$R = \{v \mid v \in H,\ B(v, v) = 0\}$$

Since the displacements of a body as a rigid entity are determined by not more than six parameters, the subset R is finite-dimensional. We separate in R the subset R^* of neutral rigid displacements

$$R^* = \{v \mid v \in R,\ F(v) = 0\}$$

and represent the subset R as a direct sum of R^* and of the orthogonal complement R_1

$$R = R^* \oplus R_1$$

Let Q, Q', Q_1 be operators of orthogonal projection of H onto R, R^* and R_1, respectively. We also introduce operators $P = I - Q$, $P' = I - Q'$, where I is the identity operator.

Since the admissible set V does not, generally, contain a zero element, hence a direct application of the Lions– Stampacchia theorem [11], as was done in [3], is not possible here. In formulating conditions of solution existence we use the subsidiary set V_0 defined as follows:

$$V_0 = \bigcap_{m,n} V^{(mn)} \bigcap_k V_{c_0}^{(k)} \bigcap_k V_{u_0}^{(k)}$$
$$V_{c_0}^{(k)} = \{v \mid v \in H,\ v(x) \operatorname{grad} \Psi_k(x) \geqslant 0, \quad \forall x \in \Gamma_k{}^c\}$$
$$V_{u_0}^{(k)} = \{v \mid v \in H,\ v(x) = 0, \quad \forall x \in \Gamma_k{}^u\}$$

Let $u_0 \in V$ satisfy the conditions

$$u_0^{(m)}(x) = u_0^{(n)}(x), \qquad \forall x \in \Gamma_{mn}$$
$$\Psi_k(x) + u_0(x) \operatorname{grad} \Psi_k(x) = 0,\ \forall x \in \Gamma_k{}^c$$
$$u_0(x) = U^{(k)}(x), \quad \forall x \in \Gamma_k{}^u \quad (k, m, n = 1, 2, \ldots, N)$$

It is possible to show that for any $u \in V$ the element $u - u_0$ belongs to V_0 and, conversely, for any $w \in V_0$ the element $w + u_0$ belongs to V. Note that V_0 contains a zero element.

The set V_0 can be obtained from V by shifting by the element u_0, hence that set is also convex and closed.

L e m m a 1. The functional $B(v, v)$ is semicoercive, i.e.

$$B(v, v) \geqslant c \| Pv \|^2, \quad \forall v \in H \tag{3.1}$$

where c is some constant independent of v. The proof of semicoercivity of $B(v, v)$ was given in [13] in the case of a single body, and can be directly extended to systems of bodies.

L e m m a 2. If $\{u_n\}$ is the minimizing sequence for the functional $J(v)$ and $F(Qu_n) \leqslant 0$, the sequence $\{\Pi(u_n)\}$ is bounded.

P r o o f. We represent $J(u_n)$ in the form

$$J(u_n) = {}^1/_2\, B(u_n, u_n) - \Pi(u_n) - F(Qu_n) - F(Pu_n)$$

Using (1.11) and taking into account the conditions of the lemma, we obtain

$$J(u_n) \geqslant {}^1/_2\, \alpha B(u_n, u_n) - F(Pu_n)$$

According to Lemma 1 there exists a $\beta > 0$ such that

$$J(u_n) \geqslant \beta \| Pu_n \| - \| F \| \| Pu_n \| \tag{3.2}$$

Let the sequence $\{\Pi(u_n)\}$ be unbounded. Then it is possible to separate in $\{u_n\}$ a sequence $\{u_{n_k}\}$ such that $\Pi(u_{n_k}) \to +\infty$, as $n \to \infty$. Since it is possible to find a $\gamma > 0$ such that

$$\Pi(v) \leqslant {}^1/_2\, B(v, v) \leqslant \gamma \| Pv \|^2, \quad \forall v \in H$$

hence $\| Pu_{n_k} \| \to +\infty$, and from (3.2) follows that $J(u_{n_k}) \to +\infty$, which means that sequence $\{u_n\}$ is not a minimizing one.

L e m m a 3. The functional ${}^1/_2\, B(v, v) - \Pi(v)$ is weakly lower semicontinuous on H. The proof is based on Kazimirov's theorem on weak semicontinuity of integral functionals [14, 15].

L e m m a 4. If for any $r \in V_0 \cap R^*$ there exists a $-r \in V_0 \cap R^*$, the set $P'[V_0]$ is weakly closed.

P r o o f. Let $r \in V_0 \cap R^*$. It can be shown that when $u \in V_0$, then $u + r \in V_0$. But by the condition of lemma $-r \in V_0 \cap R^*$, hence $u + (-r) \in V_0$. Any element $v \in P'[V_0]$ is of the form $v = u + (-r)$, where $r \in V_0 \cap R^*$. This implies that $P'[V_0] \subset V_0$ and, consequently, $P'[V_0]$ is closed. Note that the set $P'[V_0]$ is convex. The convexity and closure of $P'[V_0]$ imply its weak closure [16].

T h e o r e m 1. If $F(r) \leqslant 0$ for all $r \in V_0 \cap R$ and $F(r) = 0$ only when $-r \in V_0 \cap R$, the solution of variational problem (2.6) exists.

P r o o f. We use the Fichera's scheme of linear one-sided problems [13].
Let $J_0 = \inf_{v \in V} J(v)$ and $\{u_n\}$ be the minimizing sequence. We introduce sequence $\{z_n\}$ as follows: $z_n = u_n - u_0$ and $z_n \in V_0$. The proof consists of two stages. First, we prove the possiblity of separating in sequence $\{P'z_n\}$ a bounded sequence and, then, establish the weak convergence of some subsequence $\{u_n\}$ to the element $u \in V$ for which $J(u) = J_0$.

We represent $B(z_n, z_n)$ in the form

$$B(z_n, z_n) = 2J(u_n) - 2J(u_0) + 2F(z_n) - 2B(u_0, z_n) + 2\Pi(u_n) - 2\Pi(u_0) \tag{3.3}$$

Using the semicoercivity of $B(v, v)$ (Lemma 1) and the boundedness of $\Pi(u_n)$ (Lemma 2), we obtain from (3.3) the inequality

$$c \| Pz_n \|^2 \leqslant c_1 + 2F(z_n) - 2B(u_0, z_n) \tag{3.4}$$

Let $\{P'z_n\}$ not contain a bounded subsequence. It is, then, possible to construct sequence $t_n = \|P'z_n\|$ such that $\lim_{n\to\infty} t_n = +\infty$. We introduce sequence $w_n = z_n / t_n$. It follows from (3.4) that $\lim_{n\to\infty} \| Pw_n \| = 0$. Taking into account that $\| P''w_n \| = 1$ and Q_1 is the projection operator normal to P, we obtain

$$\| Pw_n \|^2 + \| Q_1 w_n \|^2 = \| P'w_n \|^2 = 1$$

Since $\{Q_1 w_n\}$ belongs to finite-dimensional subspace R, it is possible to extract from $\{w_n\}$ such subsequence $\{w_{n_k}\}$ that $\{Q_1 w_{n_k}\}$ converges in norm H to some $r \in R_1$, and $\| r \| = 1$. Using the conditions of the theorem it is possible, as in [13], to show that for fairly large n $F(Q_1 w_n) < 0$. From (3.4) we have

$$ct_n \| Pw_n \|^2 \leqslant c_1 / t_n + 2F(Pw_n) + 2F(Q_1 w_n) - 2B(u_0, Pw_n) \tag{3.5}$$

We have a contradiction due to the assumption that $\{P'z_n\}$ does not contain a bounded subsequence. The left-hand side of (3.5) is nonnegative, while the right-hand side is strictly below zero commencing from some n.

Since $\{P'z_n\}$ contains a bounded subsequence, it is possible to separate from $\{z_n\}$ a subsequence $\{z_{n_k}\}$ such that $\{P'z_{n_k}\}$ weakly converges to some element $P'z$, and since according to Lemma 4 $P'[V_0]$, $P'z \in P'[V_0]$ is weakly closed. Taking into account the weak lower semicontinuity of functional $1/2\, B(v, v) - \Pi(v)$ (Lemma 3), we obtain

$$J_0 = \lim J(z_{n_k} + u_0) = \lim J(P'z_{n_k} + u_0) \geqslant \underline{\lim}\, J(P'z_{n_k} + u_0) \geqslant J(P'z + u_0) = J(u) \geqslant J_0, \quad n \to \infty$$

from which we conclude that there exists an element $u \in V$ such that $J(u) = \inf_{v\in V} J(v)$.

T h e o r e m 2. If functions $W_k(x, \varepsilon_{ij})$ $(k = 1, 2, \ldots, N)$ are strictly convex with respect to ε_{ij}, the solution of the variational problem (2.6) is determined with an accuracy to neutral rigid displacements; if u is a solution of problem (2.6), any other solution w can be represented in the form $w = u + r$, where $r \in R^*$.

P r o o f. If two solutions u and v of the variational problem exist, each of them satisfy the variational inequality (2.7), hence

$$X(W(u), v - u) \geqslant F(v - u), \quad X(W(v), u - v) \geqslant F(u - v)$$

Adding the inequalities we obtain

$$X\,(W\,(u) - W\,(v),\, v - u) \geqslant 0$$

Let us assume that almost everywhere $\varepsilon'_{ij} \neq \varepsilon_{ij}$. The strict convexity of $W_k(x, \varepsilon_{ij})$ implies that

$$\int_{\Omega_k} [W_k(x, \varepsilon_{ij}') - W_k(x, \varepsilon_{ij})]\, d\Omega > \int_{\Omega_k} \frac{\partial W_k(x, \varepsilon_{ij})}{\partial \varepsilon_{ij}} (\varepsilon_{ij}' - \varepsilon_{ij})\, d\Omega$$

$$\int_{\Omega_k} [W_k(x, \varepsilon_{ij}) - W_k(x, \varepsilon_{ij}')]\, d\Omega > \int_{\Omega_k} \frac{\partial W_k(x, \varepsilon_{ij}')}{\partial \varepsilon_{ij}} (\varepsilon_{ij} - \varepsilon_{ij}')\, d\Omega$$

hence

$$X\,(W\,(u) - W\,(v),\, v - u) < 0$$

We have a contradiction related to the assumption that $\varepsilon_{ij}' \neq \varepsilon_{ij}$.

The solution $u \in V$ of the variational problem (2.6) satisfies the specified kinematic conditions on surfaces Γ_{mn}, Γ_k^u, Γ_k^c. Using the methods of variational inequalities [11] with additional assumptions on the existence of second derivatives of solution, it is possible to show that the solution also satisfies the equations of equilibrium and static conditions on surfaces Γ_{mn}, Γ_k^X, Γ_k^c, in other words, the solution of the variational problem yields a generalized solution of the problem stated in Sect. 1.

Using the variational formulation we propose numerical methods of solving contact problems for laminated bodies. In conjunction with the method of finite elements [17] the variational problem (2.6) is reduced to that of minimizing functions of many variables on a convex closed subset of finite-dimensional space. The theorems on the existence and uniqueness of solution proved in Sect. 3 are directly transferred to such problem of nonlinear programming. Use of the variational inequality (2.7) makes possible the reduction of the question of convergence of the finite element method in problems involving separation to the investigated problems of convergence in classical problems [17].

Sets of programs in FORTRAN Language have been developed for solving two- and three-dimensional contact problems of determination the stress-strain state of layered foundations. The problem of nonlinear programming is carried out using the proposed in [8] variant of the method of possible directions [16]. Some of the known solutions of problems of linearly elastic foundations are used for estimating the errors of approximate solutions. The comparison of basic characteristics (contact pressure, dimensions of contact area) show a good agreement, also in solutions with singularities.

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